

A q -Gaussian Spatial Filtering

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Abstract—The Gaussian functions have been used in several applications in Digital Image Processing area, especially as kernel in different kinds of filters through the convolution procedures. However, the Gaussian functions has been used just as representation of approximated values. There is a wide range of applications using images being researched in non-extensive statistic area defined by Constantino Tsallis, through the so-called q -entropy. Some works have shown that the physical interactions relations observed in images may be better explained under non-extensive perspective. In this paper, we present a new generalization of the Gaussian distribution through the q -Gaussian Spatial function, based on q -algebra. Also, we use this function to apply in new filters to reduce noises in the pre-processing step for image segmentation by q -entropy.

I. INTRODUCTION

In Digital Image Processing several Gaussian applications are known, however some phenomena may not be well represented by traditional Gaussian function. The relations between interactions observed in images may be better explained from the non-extensive perspective. In previous work [1], a generalization of the Gaussian function based on q -entropy was proposed, through the called q -Gaussian function. This new formalism revitalized the interest in studies from the Gaussian functions in several areas, now in the non-extensive mechanical statistic. Thus, a new perspective was observed, with great accuracy in image applications. The methodologies from segmentation may use image filtering with spatial processing. This work introduce a q -Gaussian Spatial function based on q -algebra, as a generalization of the traditional Gaussian Spatial function, with applications in image filters used in the pre-processing step for image segmentation techniques.

This new formalism cooperates for the q -algebra and defines a generalization of the traditional Gaussian Spatial function. It is called q -Gaussian Spatial function, based in non-extensive statistic by Tsallis. From this formalism, new contributions in the Digital Image Processing area may be tested through the new tools, such as creation of the image filters from the q -Gaussian Spatial function.

A. Related Works

Tsallis proposed a new technique from statistical mechanics to approach the results of physical calculation with greater accuracy, provided that it is considered as non-extensive phenomenon. Initially, a new equation was proposed in [2], known as q -entropy that generalizes the traditional Boltzmann-Gibbs-Shannon entropy, known as BGS entropy (it had its fundamentals by Shannon in [3]). The q value represents a real parameter, sometimes called parameter of non-extensivity. The

q -entropy has been very promising in several areas, replacing the BGS entropy. The q -entropy applied in Digital Image Processing area has been making important breakthroughs, such as images segmentation being one of the most studied and tested techniques, as they may be seen in previous works [4], [5], [6], [7], [8], [9], [10]. In this work, the image segmentation by q -entropy over a noiseless original image, noisy images and noisy images after filtering, is applied for comparisons.

In [11], the author describes some practical examples to apply the Gaussian Low Pass Filter widely used for smoothing images, presenting results about filtering of texts in low resolution. In another application, an image obtained by very high resolution radiometer was shown, captured by National Oceanic and Atmospheric Administration (NOAA), in order to smooth lines created by scanning sensors at the time of image capture. In the previous work [12], it was presented a survey that operates applications to digital images by traditional Gaussian filter. In this paper, new filters using the q -Gaussian Spatial function as kernel, is presented.

In [13], some studies on a possible algebra deformation related to the q -exponential and the q -logarithm function were presented. In the same work, the q -derivative and the q -integral functions were also presented. In [14], the author makes important contributions to the q -algebra making able to non-extensive formalism, specifically developing generalizations of trigonometry and hyperbolic functions, generalizations of algebra and differential calculus. The q -algebra defines generalizations about the traditional algebra, when $q \rightarrow 1$ the q -functions returns to the traditional algebra. The q -Gaussian Spatial function presented in this paper is based on q -algebra.

An interesting propose about q -Gaussian Spatial function was presented in [15], that considered a formulation as the one-dimensional symmetric q -Gaussian rotated around the origin. That formalism was based on generalized central limit theorem proposed by [16]. In that work, there was some filter applications over medical images. The results showed that the q -Gaussian Spatial filters are more effective than traditional Gaussian spatial filters.

II. TECHNICAL BACKGROUND

A. The q -Gaussian function

The q -Gaussian function, presented in [1], is a generalization of the traditional Gaussian function. The Gauss's normal distribution may be recovered as $q \rightarrow 1$. In previous work [16] for example, Tsallis defined the q -Gaussian function as

$$g_q(x) = \frac{\sqrt{\beta}}{C_q} e_q^{-\beta x^2}, \quad (1)$$

where $\beta = 1/(3 - q)$, e_q^n is a q -exponential function and C_q is a normalization constant. The q -exponential function is a generalization of the traditional exponential function e^x . The q -exponential function uses the non-extensive constant q (by Tsallis) as a deformation from traditional exponential function. The e_q^n was defined in [17] as an analog function of the e^x , given by $[1 + (1 - q)x]_+^{\frac{1}{1-q}}$ and its constraints are given by:

$$e_q^x = \begin{cases} [1 + (1 - q)x]^{\frac{1}{1-q}}, & \text{for } q \neq 1 \text{ and } 1 + (1 - q)x > 0 \\ 0, & \text{for } q \neq 1 \text{ and } 1 + (1 - q)x \leq 0 \\ e^x, & \text{for } q = 1 \end{cases} \quad (2)$$

The normalization factor C_q is also dependent on non-extensive constant q as the following:

$$C_q = \begin{cases} 2\sqrt{\pi}\Gamma(\frac{1}{1-q})\frac{1}{(3-q)\sqrt{1-q}\Gamma(\frac{3-q}{2(1-q)})}, & \text{for } -\infty < q < 1 \\ \sqrt{\pi}, & \text{for } q = 1 \\ \sqrt{\pi}\Gamma(\frac{3-q}{2(q-1)})\frac{1}{\sqrt{q-1}\Gamma(\frac{1}{q-1})}, & \text{for } 1 < q < 3 \end{cases} \quad (3)$$

The Gamma function $\Gamma(n) = (n - 1)!$ is an extension of factorial function to real or complex numbers.

The Figure 1(a) shows an example of the q -Gaussian filtering application over a one-dimensional sign. This signal is a unit impulse, with 12 elements. Its values are $f_{original} = \{0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0\}$. We add a Gaussian noise with standard deviation 0.04 over this signal. The result $f(x)$ is shown in Figure 1(b) with the following values:

$$f = \{0, 0.091, 0.068, 0.043, 0, 1, 0.075, 0.010, 0.042, 0.014, 0, 0.011\}.$$

In this example, we wanted to reduce noise from signal $f(x)$ through the filter q -Gaussian application $w_q(x)$. Thus, we used a one-dimensional q -Gaussian function given by Equation (1) with $q = 1$. The Figure 2(a) shows the graphic representation of $w_q(x)$ with the following values:

$$w_q(x) = \{0.0001, 0.0044, 0.0540, 0.2420, 0.3989, 0.2420, 0.0540, 0.0044, 0.0001\}$$

In the next step, the convolution was made between the input signal $f(x)$ and the q -Gaussian filter $w_q(x)$, represented by $g(x) = f(x) * w_q(x)$. After that, the result is an output filtered signal normalized by the maximum value of input signal, represented in Figure 2(b) with its values given by:

$$g(x) = \{0, 0, 0, 0, 0, 0, 0, 0.018, 0.359, 1, 0.435, 0.089, 0.12, 0.045, 0.035, 0.091, 0.033, 0.002, 0\}$$

Finally, the Figure 2(c) shows the divergence between the original signal $f_{original}$ and the output filtered signal $g(x)$.

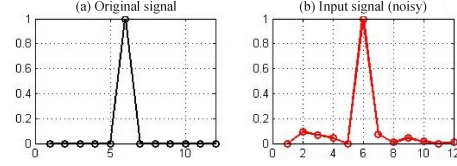


Figure 1: Example of one-dimensional signal filtering by convolution, from q -Gaussian function: (a) Original sign $f_{original}$ used as example. (b) The same signal after application of noise $f(x)$.

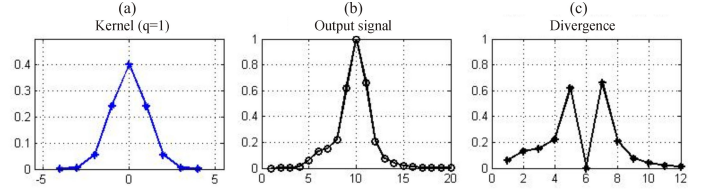


Figure 2: The results of example of one-dimensional signal filtering by convolution: (a) $w_q(x)$: q -Gaussian kernel with $q = 1$ and window size $m = 9$. (b) $g(x)$: output signal, filtered by convolution from one-dimensional q -Gaussian function. (c) Divergence between the original signal $f_{original}$ and the output filtered signal $g(x)$.

B. Image Segmentation by q -Entropy

The q -entropy was formulated by Tsallis [2]. It is a generalization of the standard BGS entropy by Boltzmann-Gibbs-Shannon [3]. That traditional BGS entropy does not be able to represent physical systems with non-extensive effects. The BGS entropy is defined as

$$S_q(P) = \frac{1 - \sum_{i=1}^k (p_i)^q}{q - 1}, \quad (4)$$

where p_i is the probability of physical system to be in i state, being $i = [1, 2, \dots, k]$. The k value is a range of system possibilities and the q is the non-extensibility degree. If the $q \rightarrow 1$, then the q -entropy converges to BGS entropy. An important property of q -entropy is called the pseudo-additivity, defined as

$$S_q(P, Q) = S_q(P) + S_q(Q) + (1 - q)S_q(P)S_q(Q). \quad (5)$$

The image segmentation from thresholding method by q -entropy was presented in [4], being based on making two-dimensional histogram from the original image acquired with n grayscales. Also, regarding to the distribution of probabilities of the grayscale given by $P = \{p_1; p_2; \dots; p_n\}$, separating P into two normalized distributions representing two different places, P_P and P_Q . Compute the q -entropy according to Equation (4), being,

$$S_q(P) = \frac{1 - \sum_{i=1}^t (\frac{P_i}{P_P})^q}{q - 1} \quad (6)$$

and

$$S_q(Q) = \frac{1 - \sum_{i=t+1}^n (\frac{P_i}{P_Q})^q}{q - 1}. \quad (7)$$

Then, the pseudo-additivity by Tsallis through the Equation (5) is calculated and this procedure from $t = t + 1$ is repeated.

Finally, the threshold chosen is the maximum value in the pseudo-additivity. Thus, the optimal threshold is given as

$$t_{opt}^{Tsallis} = \max[S_q(P, Q)]. \quad (8)$$

The maximization of the threshold is dependent on the q value, i.e., it is able to compute an optimal threshold through the value variation from q , being able to adapt to behavior of image.

III. THE q -GAUSSIAN SPATIAL FUNCTION

Considering a q -Gaussian function $g_q(x)$, where x is the intensity of an applied input signal represented on the axis x as Equation (1), given by

$$g_q(x) = \frac{\sqrt{\beta}}{C_q} e_q^{-\beta x^2}.$$

It is also able to consider, a q -Gaussian function $g_q(y)$, similar to Equation (1), representing the intensity of an applied input signal represented on the axis y , given by

$$g_q(y) = \frac{\sqrt{\beta}}{C_q} e_q^{-\beta y^2}. \quad (9)$$

A q -Gaussian Spatial function is defined by the multiplication between two q -Gaussian functions. However, $g_q(x)$ and $g_q(y)$ are based on q -algebra, and being the q value considered a non-extensive parameter as a possible deformation from the traditional algebra. Borges, in his work [13], showed that the result of product between two q -analog functions is the q -multiplication between them, defined as

$$x \otimes_q y \equiv [x^{1-q} + y^{1-q} - 1]_+^{\frac{1}{1-q}}, \quad (10)$$

for $(x, y > 0)$.

An analysis about the q -algebra, may be seen at work [16], with explanation about q -Gaussian, q -addition, q -multiplication, q -exponential, q -logarithm and q -Fourier.

Applying the q -multiplication formalism, given by Equation (10), may be considered:

$$\begin{aligned} G_q(x, y) &= g_q(x) \otimes_q g_q(y) \\ G_q(x, y) &= [g_q(x)^{1-q} + g_q(y)^{1-q} - 1]_+^{\frac{1}{1-q}} \end{aligned}$$

Finally, from this result, we introduced a definition of the q -Gaussian Spatial function as:

$$G_q(x, y) = [(\frac{\sqrt{\beta}}{C_q} e_q^{-\beta x^2})^{1-q} + (\frac{\sqrt{\beta}}{C_q} e_q^{-\beta y^2})^{1-q} - 1]_+^{\frac{1}{1-q}} \quad (11)$$

IV. METHODOLOGY

Supposing the possibility of several applications of the traditional Gaussian Spatial function, how would the accuracy of such applications using the q -Gaussian Spatial function be? How would the filtering behavior by kernel based in q -Gaussian Spatial function be? Could the convolution operator use, deformed by q parameter from q -entropy, improve digital image filtering results? The Figure 3 shows the methodology used in these experiments.

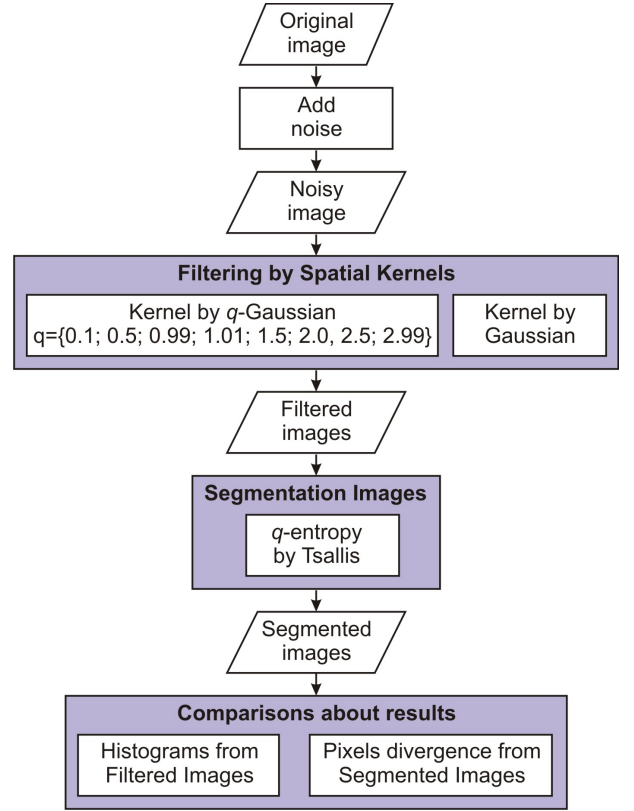


Figure 3: Methodology applied over the experiments.

A. The q -Gaussian Spatial Filtering

Proposing to use the q -Gaussian Spatial function, according Equation (11), as experiments, we applied the Gaussian noise over the original digital image and obtained 3 noisy images. From this 3 noisy images, we filtered it by convolution, using a q -Gaussian Spatial function as kernel of the filter, presented following:

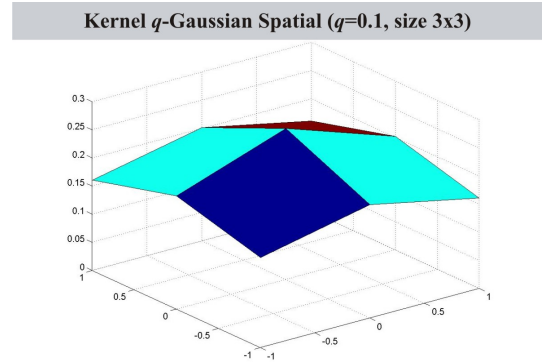


Figure 4: The kernel used in our experiments: q -Gaussian Spatial function with $q = 0.1$ and the 3×3 window size.

Several experiments were tested using values as $q = \{0.1, 0.5, 0.99, 1.01, 1.5, 2.0, 2.5, 2.99\}$, but the experiments with $q = 0.1$ were randomly chosen to show in this paper. For testing the results obtained by the filtering, we compared the

filtered images and the noisy images through their histograms. After that, the segmentation by q -entropy over all images was applied, and the results by divergence of pixels were compared.

V. EXPERIMENTS

From the original image shown in Figure 5(a), with 1027 by 768 pixels, 256 grayscale, the Gaussian noise with standard deviation as $\sigma = \{0.04, 0.2, 0.4\}$ was applied. Thus, 3 noisy images were obtained, presented in Figure 5(b), (c) e (d).

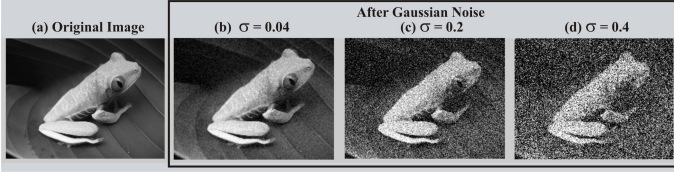


Figure 5: The original input image, no noise (a). The same image after Gaussian noise using different standard deviation: $\sigma = 0.04$ (b), $\sigma = 0.2$ (c) and $\sigma = 0.4$ (d).

Applying the image segmentation by q -entropy, using $q = 0.5$, over the original input image. The Figure 6 shows this result.

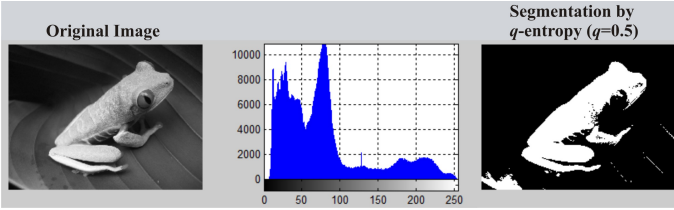


Figure 6: Original image (left). Luminance histogram of 256 grayscale (middle). Segmented image by q -entropy using $q = 0.5$ (right).

A. Segmentation from noisy image before filtering

Applying the image segmentation by q -entropy, using $q = 0.5$, over the other 3 noisy images. The Figure 7 shows the results.

The Figure 9 shows the divergence between the histograms from noisy images and the histogram from original image. Also, the percentage of the divergent pixels of each segmented noisy image was calculated, from the segmented original image. The Figure 12 shows its results.

B. Segmentation from noisy image after q -Gaussian Spatial filtering

The q -Gaussian Spatial filtering was applied over the 3 noisy images shown in Figure 5(b), (c) e (d), by convolution. The kernel of the filter used is presented in Figure 4. Thus, applying the image segmentation by q -entropy (using $q = 0.5$) over the 3 filtered images. The results in Figure 8 show that the q -Gaussian Spatial filtering was able to reduce the noise of all images, in contrast to the segmented images before filtering shown in Figure 12.

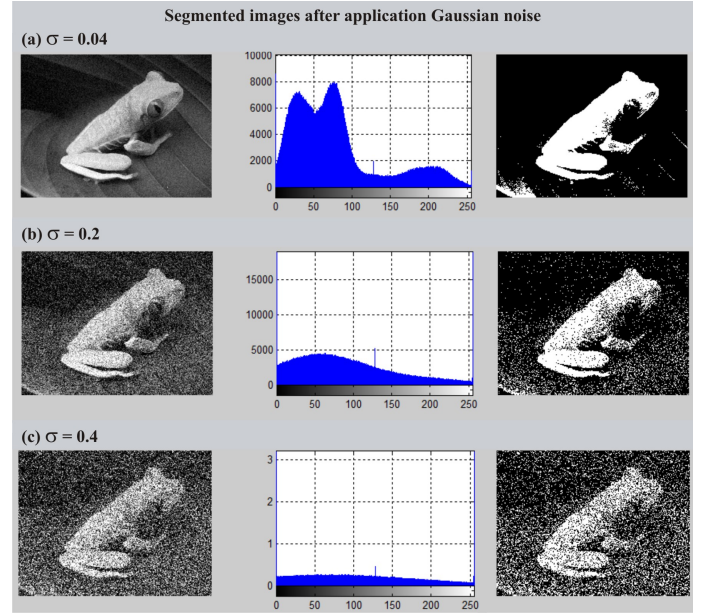


Figure 7: Noisy images before segmentation (left). Luminance histogram of 256 grayscale (middle). Segmented images by q -entropy, using $q = 0.5$ (right).

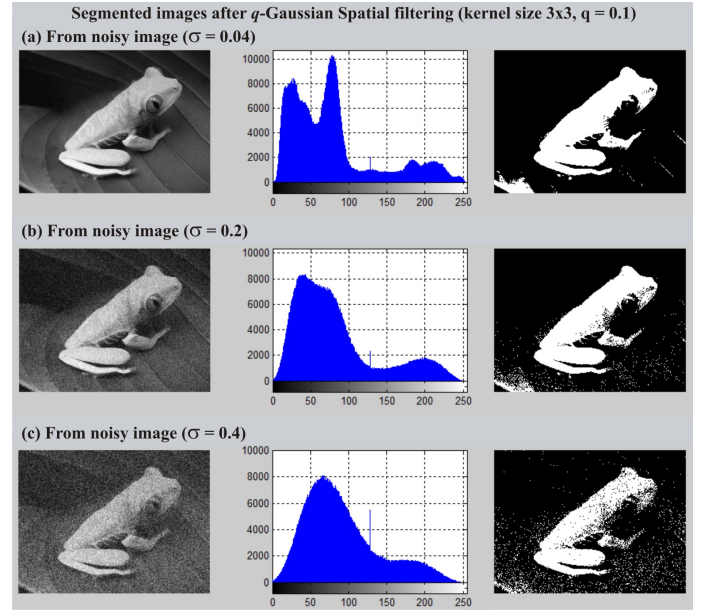


Figure 8: Filtered images (left). Luminance histogram of 256 grayscale (middle). Segmented images by q -entropy using $q = 0.5$ (right).

The Figure 11 shows the divergence between the histograms from filtered images and from original image. After that, the percentage of divergent pixels of each segmented filtered image is calculated, from the segmented original image. The Figure 14 presents these results.

C. Segmentation from noisy image after traditional Gaussian Spatial filtering

The same setup to filtering applied on experiments shown in the Figure 8 now is made by the traditional Gaussian Spatial filtering in the pre-processing step, using the 3 x 3 window. After that, the divergence between the histograms is calculated, from filtered images by traditional Gaussian Spatial function and compared to the original image histogram.

Similarly, the same setup to segmentation observed in experiments shown in Figure 14 was applied by the traditional Gaussian Spatial filtering. Thus, the divergence between the segmentation from filtered images by traditional Gaussian Spatial function and the segmentation from original image, was calculated. The Figure 13 shows these results.

VI. RESULTS AND DISCUSSION

In this work, the application of the q -Gaussian Spatial function as a filter to reduce noise image was presented, with $q = 0.1$ and 3 x 3 window, that may be seen in Figure 4. We applied the q -Gaussian Spatial filter over the 3 images that received Gaussian noise with different standard deviation. After that, these 3 noisy images were segmented by q -entropy. The segmented noisy images may be seen in Figure 7 and the segmented filtered images by q -Gaussian Spatial filter may be seen in Figure 8. Also, we applied the traditional Gaussian Spatial filter over the same 3 noisy images and these images were segmented by q -entropy.

A. The comparisons between histograms

After the application of Gaussian noise over the original image, comparisons between histograms were made. For comparing the difference of the number of pixels at each grayscale were computed between histograms. The closer the zero value, lower the divergence is. The percentage above each graph shows the divergence, and the σ values at the bottom of each graph shows the standard deviation of Gaussian noise applied on the images, in Figures 9, 10 and 11. When comparing the divergence percentage from the histograms of the same standard deviation in noise, the smaller the divergence, the greater filter efficiency is.

The divergence between histograms from the noisy images and the original image, no filtering, is shown in figure 9.

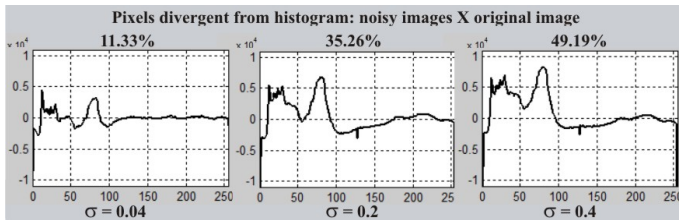


Figure 9: Divergent pixels over histogram from noisy images (no filtering) in contrast to the histogram from original image.

The differences between the histograms of noisy images after applying the traditional Gaussian Spatial filter and the histogram of the original image is computed, and the results are presented in Figure 10.

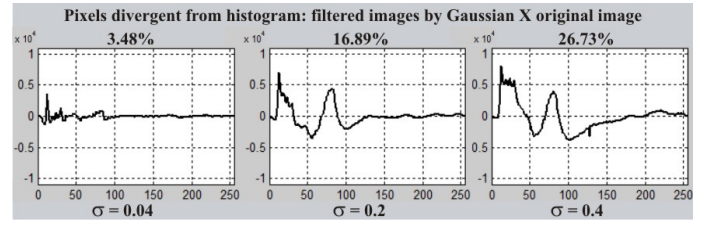


Figure 10: Divergent pixels over histograms from filtered images by traditional Gaussian Spatial in contrast to the histogram from original image.

Also, the differences between the histograms of noisy images after applying the q -Gaussian Spatial filter and the histogram of the original image is calculate, and the results may be seen in Figure 11.

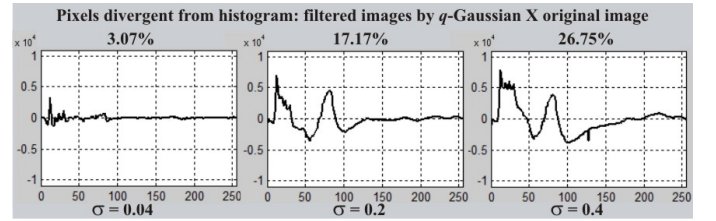


Figure 11: Divergent pixels over histogram from filtered images by q -Gaussian Spatial in contrast to the histogram from original image.

The histograms made from the filtered images have smaller divergences in comparison to noisy images. Note that it is show that the q -Gaussian Spatial filter was very effective in reducing the noise on the noisy image with $\sigma = 0.04$, presented 11.33% of different values in relation to histogram from the original image, while after applying the q -Gaussian Spatial filtering this difference was reduced to just 3.07%. Using the traditional Gaussian Spatial filter was reduced to 3.48%.

B. The comparisons between segmented images

Analysing the segmented images (the original, noisy and filtered) by q -entropy, the results may be seen respectively in 6, 7 and 8. After the segmentation, the pixel by pixel divergence between segmented images were computed. The white points represent the divergent pixels and the black points mean the same pixel value in both segmented images, in Figures 12, 13 and 14. When comparing the divergence percentage from the images of the same standard deviation in noise, the smaller the divergence, the greater will have been filter efficiency.

The divergence between the segmented original image and segmented noisy images were calculated, no filtering, and the results are shown in Figure 12.

Also, the pixel by pixel divergence between the segmented original image and the segmented filtered images by traditional Gaussian Spatial filters were calculated. These results may be seen in Figure 13.

By the same technique, the pixel by pixel divergence between the segmented original image and the segmented filtered images by q -Gaussian Spatial filters were calculated, and the results are presented in Figure 14.

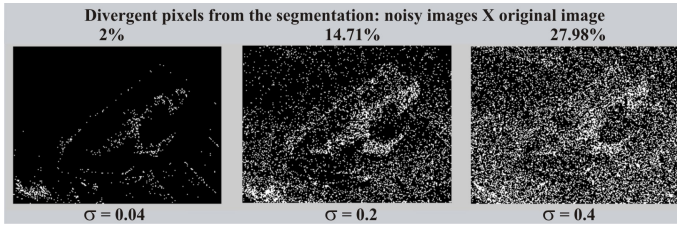


Figure 12: The white points represent the divergent pixels from segmented noisy images (no filtering) in contrast to segmented original image.

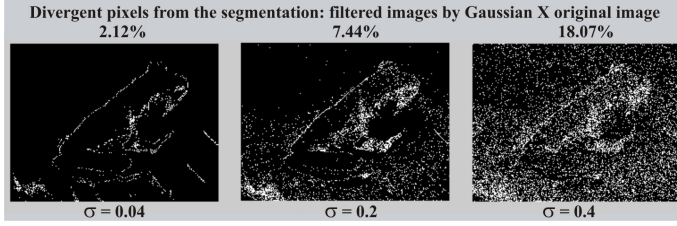


Figure 13: The white points represent the divergent pixels from segmented images after filtering by traditional Gaussian Spatial in contrast to the original image segmented.

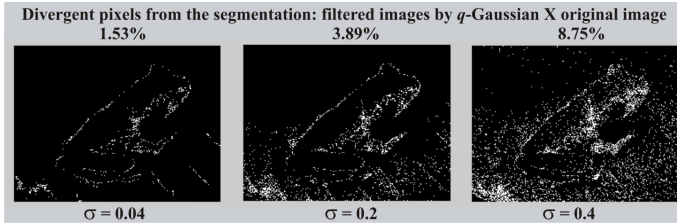


Figure 14: The white points represent the divergent pixels from segmented images after filtering by q -Gaussian Spatial in contrast to the original image segmented.

Note that the segmented images after filtering by q -Gaussian Spatial function have a lower percentage of pixel by pixel divergence in relation to the segmented original image and the segmented filtered images by traditional Gaussian filters. It is showing that the q -Gaussian Spatial filter was efficient in reduce noise as pre-processing step of segmentation by q -entropy, too.

VII. CONCLUSION

In this paper, the q -Gaussian Spatial function as a generalization of the traditional Gaussian Spatial function was presented, provided that for $q \rightarrow 1$, the q -Gaussian Spatial function reduces to it. We proved that the use of the filters with the q -Gaussian Spatial function as kernel was very efficient to reduce noise in these cases. For all q values tested, reduction of noise after filtering by q -Gaussian Spatial function and improvement of the segmented image quality were observed. These applications of the q -Gaussian Spatial function open new possibility for variation of the non-extensive parameter q , which may provide better results, and the quality in segmented images by q -entropy may be improved. We expect to extend this application in new directions in future works, by other

filters setups, different q values and window sizes, including other techniques of image segmentation by entropy.

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